A theory of thin airfoils and slender bodies in fluids of finite electrical conductivity with aligned fields

By EDMUND C. LARY†

Cornell University, Ithaca, New York

(Received 1 April 1960 and in revised form 24 February 1961)

The steady incompressible flow of a non-viscous conducting fluid about thin airfoils and slender bodies is studied for the case of a uniform applied magnetic field aligned with the undisturbed fluid stream. Solutions are found, subject to the restriction of small perturbations to the applied field. This condition determines an upper limit on the range of conductivity to which the solutions are applicable. Certain results for larger values of conductivity are presented and discussed.

The lift on airfoils is calculated, including the possibility of magnets and externally driven currents inside the airfoil, and a magnetohydrodynamic analogue to the Kutta condition is discussed. Drag formulae are presented for airfoils and slender bodies, and the distribution of internal currents and magnets for zero drag is shown. Optimum drag airfoils and bodies are discussed briefly.

Introduction

The magnetohydrodynamic flow about thin airfoils in an incompressible fluid of high electrical conductivity has been treated by Sears & Resler (1959) for both crossed and aligned magnetic and velocity fields. The case of crossed fields has been extended recently by McCune (1960) to apply to values of magnetic Reynolds number in excess of unity.

The present paper is concerned with an extension of the aligned-fields analysis to arbitrary values of conductivity and to the flow about slender bodies of revolution. Singular solutions of the linearized equations for incompressible flow are derived and employed in the solution of boundary-value problems appropriate to thin-airfoil and slender-body flows. As usual, viscous effects are neglected throughout, except that the usual Kutta condition is applied wherever appropriate for lifting airfoils.

In the limit of small conductivity, the magnetic field interacts only weakly with the flow, and the solutions must reduce to the conventional hydrodynamic results. On the other hand, for very large conductivity the perturbations are large in the magnetic boundary layer (Sears & Resler 1959), so that the linearization is valid only outside this layer. The latter consideration implies a restriction on the range of conductivity for which the linearized analysis is applicable to the entire flow field. It will be shown that the restriction is that the thickness of the

† Now with United Aircraft Corporation Research Laboratories, East Hartford, Connecticut.

magnetohydrodynamic interaction region at the body (or magnetic boundary layer) be greater than the thickness of the body. On the other hand, from conventional boundary-layer theory, the neglect of viscosity is certainly justified if the thickness of the viscous boundary layer is small compared to the body thickness. Examples of situations in which these conditions are satisfied are: (1) in mercury, $R_m = 1{\cdot}3$ and $R_e = 10^7$, based on a length of 10 cm and a speed of 10 m/sec; and (2) in sea water, $R_m = 0{\cdot}01$ and $R_e = 10^9$, based on 100 m and 20 m/sec.

The basic equations

The governing equations for the steady flow of an inviscid, incompressible fluid of finite conductivity have been given by Sears & Resler (1959). By employing the assumption of small perturbations to uniform applied velocity and magnetic fields, both parallel to the x-axis, they obtain the following linearized equations:

$$\nabla \times \mathbf{j} = R_m\left(\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial \mathbf{h}}{\partial x}\right), \tag{1}$$

$$\boldsymbol{\omega} = \alpha^2 \mathbf{j}, \tag{2}$$

$$p = -v_x. \tag{3}$$

Here, MKSQ units are implied and uniform permeability μ and scalar conductivity σ are assumed. The fluid velocity $\mathbf{V}$ and magnetic field $\mathbf{H}$ have been replaced by the non-dimensional perturbation vectors $\mathbf{v} = (\mathbf{V}-\mathbf{V}_0)/V_0$ and $\mathbf{h} = (\mathbf{H}-\mathbf{H}_0)/H_0$, respectively, and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and $\mathbf{j} = \nabla \times \mathbf{h}$ are the corresponding vorticity and current density vectors. The subscript 0 denotes uniform values at infinity, and p is the pressure perturbation $(P-P_0)/\rho V^2$ in non-dimensional form. The co-ordinates of physical space are normalized to L, and R_m is the corresponding magnetic Reynolds number $\mu\sigma V_0 L$. The ratio of magnetic-energy density to the fluid kinetic-energy density, $\mu H_0^2/\rho V_0^2$, is denoted by α^2, so that $1/\alpha$ is the ratio of fluid speed to the Alfvén speed for infinite conductivity, or the 'Alfvén number' of the flow (Sears 1959).

The curl of equation (1) is combined with equation (2) to yield the governing differential equation for current and vorticity,

$$(\nabla^2 - 2k\,\partial/\partial x)\,\mathbf{j} = 0. \tag{4}$$

This may be recognized as the Oseen equation of linearized viscous flow (e.g. Lamb 1932), where $2k = R_m(1-\alpha^2)$ has replaced the conventional Reynolds number R_e. In general, solutions to the above equations are subject to the usual conditions of solenoidal $\mathbf{v}$ and $\mathbf{h}$ and to Ohm's Law (which appears in a differential form as equation (1)),

$$\mathbf{J} = \sigma(\mathbf{E} + \mu\mathbf{V}\times\mathbf{H}). \tag{5}$$

The Maxwell equation $\nabla \times \mathbf{E} = 0$ completes the set of equations, which should suffice in principle for the solution of three-dimensional boundary-value problems consistent with the assumption of small perturbations.

For the case of plane and axisymmetric flows to be treated herein, the fact that all perturbations vanish at infinity (a point which will be discussed in a subsequent section) leads to the conclusion that $\mathbf{E} = 0$. The linearized form of equation (5) is therefore

$$j_z = R_m(h_y - v_y), \tag{6}$$

for plane flow. The same result applies to axisymmetric flow with the subscripts θ and r replacing z and y, respectively.

The general solution to equation (2) for the perturbations $\mathbf{v}$ and $\mathbf{h}$ is obtained by direct integration, and is

$$\mathbf{v} = \alpha^2\mathbf{h} + \mathbf{a}_1, \tag{7}$$

where $\mathbf{a}_1$ is an arbitrary irrotational and solenoidal vector field which vanishes at infinity. If $\mathbf{v}$ and $\mathbf{h}$ are separated into irrotational and rotational parts, denoted respectively by the subscripts 1 and 2, then the irrotational part must satisfy the condition

$$\mathbf{v}_1 = \mathbf{h}_1, \tag{8}$$

and the rotational part may be assumed to have the form

$$\mathbf{v}_2 = \alpha^2\mathbf{h}_2, \tag{9}$$

with no loss in generality.

Singular solutions

The irrotational components satisfy Laplace's equation in the stream function and potential. The singular solutions are the familiar source and, for plane flow, vortex. The rotational components satisfy, according to equations (6) and (9), relations of the form

$$j_{z_2} = 2kh_{y_2}, \tag{10}$$

which is Oseen's equation when written in terms of the stream function. The potential does not exist for the rotational solutions, but equation (10) is satisfied identically for plane flow by a pseudo-potential. In either case, singular solutions of Oseen's equation lead to the 'rotational source' and 'rotational vortex' for plane flow,

$$\mathbf{h}_2 = \alpha^{-2}\mathbf{v}_2 = \pm\tfrac{1}{2}\pi^{-1}k\,e^{k(x-\xi)} \times\{(x-\xi)\,d^{-1}K_1(\pm kd) \pm K_0(\pm kd), (y-\eta)\,d^{-1}K_1(\pm kd), 0\}, \tag{11}$$

$$\tilde{\mathbf{h}}_2 = \alpha^{-2}\tilde{\mathbf{v}}_2 = \pm\tfrac{1}{2}\pi^{-1}k\,e^{k(x-\xi)} \times\{(\eta-y)\,d^{-1}K_1(\pm kd), (x-\xi)\,d^{-1}K_1(\pm kd) \mp K_0(\pm kd), 0\}, \tag{12}$$

where K_n denotes the modified Bessel function of the second kind and nth order, $d = \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}}$ (and $s = \{(x-\xi)^2 + r^2\}^{\frac{1}{2}}$ below), and the sign choice $\pm$ refers to $k > 0$ and $k < 0$, respectively. The rotational source for axisymmetric flow is given by

$$\hat{\mathbf{h}}_2 = \alpha^{-2}\hat{\mathbf{v}}_2 = \tfrac{1}{4}\pi^{-1}s^{-3}\,e^{k(x-\xi\mp s)}\{(x-\xi)(1\pm ks) + ks^2, r(1\pm ks), 0\}. \tag{13}$$

In each case, the rotational singular solutions approach the corresponding irrotational solutions at very small distances, and also in the limit of small $|k|$ (i.e. $R_m \ll 1$ or α^2 near unity). On the other hand, the rotational solutions are dominated by exponential convergence to zero at large distances, everywhere except in a parabolic region with focus at the singular point. The region inside the parabolic surface will be referred to as the 'wake' (e.g. Lamb 1932). Evidently the magnetohydrodynamic wake will appear upstream of the singular point for

$\alpha^2 > 1$ and downstream for $\alpha^2 < 1$, as noted by Greenspan & Carrier (1959) for viscous flow of a conducting fluid over a finite flat plate with aligned fields. In general, the irrotational solutions apply to disturbances which fully penetrate the fluid, whereas rotational solutions involve a 'shielding' of the disturbance by the fluid, tending to limit its effect to a wake region.

Boundary conditions

All disturbances to the uniform conditions $\mathbf{H}_0$, $\mathbf{V}_0$ which are due to a finite body (in either two- or three-dimensional flow) vanish at infinity in a fluid of zero conductivity. This statement applies as well to the case of finite conductivity because of the existence of a dissipative mechanism which attenuates disturbances created at the body. It has been shown by Sears & Resler (1959) that, in the limit of infinite conductivity, the assumption of small perturbations which vanish at infinity leads to identical perturbations in $\mathbf{v}$ and $\mathbf{h}$, which are exactly those of classical hydrodynamic flow about the same body. Stewartson (1960) has pointed out that other solutions are also possible in this limit. The present paper applies to the small R_m situation, which should be expected to represent a perturbation to the familiar hydrodynamic solutions. The fact that perturbations may be small in this limit everywhere in the flow, i.e. that the boundary conditions for zero viscosity and small R_m are consistent with small perturbations even at the boundary, leads to a unique solution of the equations.

The linearized equations were obtained by retaining only first-order perturbation quantities. It is consistent to approximate the boundary conditions at the body to the same order. For definiteness, a family of bodies of length L is considered in both plane and axisymmetric flow. In terms of the non-dimensional variables, these bodies lie in the range [0, 1] of x and may be expressed in terms of the parameter ϵ. If Y_0 and R_0 are shape functions of order unity, then, on the body surface,

$$y = Y(x) = \epsilon Y_0(x) \quad \text{and} \quad r = R(x) = \epsilon R_0(x). \tag{14}$$

In conventional hydrodynamic flow, the order of magnitude of the velocity perturbations (with the possible exception of stagnation regions near the leading- and trailing-edges) is ϵ, and ϵ is assumed small compared to unity. The same is true of the present analysis, except that the assumption that perturbations are small will be consistent only subject to certain restrictions on the parameters R_m and α^2.

The boundary condition on $\mathbf{V}$ is approximated by neglecting v_x compared to unity, thus yielding the familiar results, correct to first order in ϵ,

$$v_y(x, Y_{u,l}(x)) = Y'_{u,l}(x) \quad \text{and} \quad v_r(x, R(x)) = R'(x). \tag{15}$$

The subscripts u and l denote the upper and lower airfoil surfaces, respectively.

The situation is not quite so straightforward for the condition on $\mathbf{H}$, since a functional relationship exists between the normal and tangential components of $\mathbf{H}$, and the boundary conditions express neither as an explicit function of x. In particular, $\mathbf{J}$ vanishes inside the body, independent of its conductivity, so that $\mathbf{H}$ is the gradient of a harmonic function. On the other hand, consistent with the assumption of small perturbations in the flow, it may be assumed that $\mathbf{h}$, and hence $\partial\mathbf{h}/\partial x$, is of order ϵ inside bodies as well. By integrating the divergence and curl relations over the body cross-section, it may be shown directly that h_x and h_y are continuous to first order in ϵ across an airfoil, and that h_r vanishes at the surface of a body of revolution. These results do not, of course, apply to the infinite-conductivity case of Sears & Resler, where $\mathbf{h}$ is large both at the boundary and inside the body.

A degree of generality may be added by considering the possibility of (1) a (non-dimensional) current distribution $I(x)$ in the z-direction inside airfoils, and (2) a (non-dimensional) magnetic source distribution $M(x)$ inside either the slender bodies or airfoils. Condition 1 violates the requirement that $\mathbf{E} = 0$, but may serve as a good approximation if the airfoil contains wires impressed with a very small electric field, especially if the conductivity of the wire is great compared to that of the fluid. Condition 2 is not in conflict with the general statement $\nabla . \mathbf{B} = 0$, but is rather a statement of the non-uniformity of μ associated with magnetization effects in the body. $I(x)$ and $M(x)$ are written, respectively, as the integrals of $\nabla \times \mathbf{h}$ and $\nabla . \mathbf{h}$ over the body cross-section at any station x. The continuity conditions on $\mathbf{h}$ discussed above are replaced by the conditions

$$\left.\begin{array}{ll} & \Delta h_y(x) = M(x), \quad \Delta h_x(x) = -I(x), \\ \text{and} & h_r\{x, R(x)\} = M(x)/2\pi R(x), \quad h_x\{x, R(x)\} = h_x(x, 0), \end{array}\right\} \tag{16}$$

for plane and axisymmetric flow, respectively. The difference of a quantity evaluated across the airfoil has been denoted by Δ, where, for example,

$$\Delta h_y(x) = h_y\{x, Y_u(x)\} - h_y\{x, Y_l(x)\}. \tag{17}$$

It is evidently necessary that I and M be of order ϵ or less for plane flow and M of order ϵ^2 or less for axisymmetric flow.

Thin airfoil theory

The boundary condition on $\mathbf{V}$ in thin-airfoil flow may be expressed in terms of camber $C(x)$ and thickness $T(x)$, defined by

$$Y_{u,l}(x) = C(x) \pm \tfrac{1}{2}T(x). \tag{18}$$

Here camber includes the effect of incidence of the airfoil. Equation (15) becomes

$$\Delta v_y(x) = T'(x), \tag{19}$$

so that camber effects, if present, affect only Δv_x (and not Δv_y) to first order in ϵ. This condition, as yet unspecified, is expressed in the form

$$\Delta v_x(x) = -G(x), \tag{20}$$

where it is expected that $G(x)$ will depend in some way upon $C(x)$. In the conventional lifting-airfoil theory G is the vortex distribution, and it is the non-dimensional pressure loading, by equation (3), even in the present theory.

Another step is necessary in order to linearize the boundary conditions for thin airfoils. The above relations are the approximate conditions which must be satisfied at the airfoil. It must be demonstrated in addition that it is sufficient to satisfy the same conditions at the x-axis. It is well known that this step is valid for conventional airfoil theory, to which the present theory must reduce for $R_m = 0$, and therefore should be valid for sufficiently small R_m. (An upper bound on the range of k is established in the approximate solutions of the boundary-value problem.) This step allows the thickness and lifting problems of conventional thin-airfoil theory to be studied separately, and an analogous decomposition is possible in the case of a conducting fluid. The two composite problems, thickness-plus-magnetization and camber-effects-plus-current, will be referred to in a more general sense as the thickness and lifting problems of finite-conductivity thin-airfoil theory.

The thickness problem

An airfoil of thickness $T(x)$ with $C(x) = 0$ is assumed to contain magnetization per unit span $M(x)$ made non-dimensional by H_0. The boundary conditions at infinity and on the airfoil surface may be satisfied by distributions $f_1(x)$ and $f_2(x)$, respectively, of irrotational and rotational source solutions, singular on the x-axis, leading to the following integral equations:

$$\pm\tfrac{1}{2}\pi^{-1}(1-\alpha^2)\int_0^1 f_2(\xi)\,(yd^{-1})\,k\,e^{k(x-\xi)}K_1(\pm kd)\,d\xi = \tfrac{1}{2}\{M(x)-T'(x)\} \quad (21)$$

and

$$\tfrac{1}{2}\pi^{-1}(1-\alpha^2)\int_0^1 f_1(\xi)\,(yd^{-2})\,d\xi = \tfrac{1}{2}\{T'(x)-\alpha^2 M(x)\}. \quad (22)$$

The latter equation has the form of the familiar integral equation of conventional airfoil theory, for which the solution is

$$\frac{1}{2\pi}\int_0^1 f_1(\xi)\,(y\,d^{-2})\,d\xi = \tfrac{1}{2}f_1(x)+O\,(\epsilon^2 f_1). \quad (23)$$

(This result is obtained by expanding $f_1(\xi)$ in a Taylor series about $\xi = x$ and observing that $y = \frac{1}{2}T(x)$ is of order ϵ.) The above solution is valid except near blunt edges, where the linearized equations are also invalid, and is subject to the usual restriction that $\epsilon \ll 1$.

The integral involving f_2 may be written in the form

$$\pm\frac{1}{2\pi}\int_0^1 f_2(\xi)\,k\,e^{k(x-\xi)}\,(y\,d^{-1})\,K_1(\pm kd)\,d\xi = \tfrac{1}{2}f_2(x)-\tfrac{1}{2}yQ\pi^{-1}+O(\epsilon^2 f_2), \quad (24)$$

where the result of equation (23) has been used and where

$$Q = \int_0^1 f_2(\xi)\,\{d^{-2}\mp k\,d^{-1}\,e^{k(x-\xi)}K_1(\pm kd)\}\,d\xi. \quad (25)$$

The following bound on the magnitude of Q may be obtained for large $|k|$ (Lary 1959) by employing the asymptotic expansions of K_n:

$$|Q| \leqslant |f_2|\,[x^{-1}+(1-x)^{-1}+(2\pi\,|k|)^{\frac{1}{2}}\{x^{-\frac{1}{2}}+(1-x)^{-\frac{1}{2}}\}\{1+O(k^{-1})\}]. \quad (26)$$

Consequently, the remainder terms in equation (24) are small except at the leading and trailing edges if $\epsilon\sqrt{|k|} \ll 1$, that is, if

$$|k| \ll \epsilon^{-2}. \quad (27)$$

The fact that $|k|$ may be large compared to unity justifies *a posteriori* the use of asymptotic-expansion formulae in obtaining equation (26), since the bound on Q increases monotonically with $|k|$. The limitation on k ensures not only that the perturbations are small, but also that they approach the irrotational solution very near the body.

Subject to the restrictions on the magnitude of ϵ and k, the approximate distribution functions for singular solutions on the x-axis are therefore

$$f_1(x) = \frac{T'(x)-\alpha^2 M(x)}{1-\alpha^2}, \quad f_2(x) = \frac{M(x)-T'(x)}{1-\alpha^2}. \tag{28}$$

Solution for flow perturbations. The solution of the thickness problem now consists of calculating the perturbation quantities associated with the singularity distributions derived above. For example,

$$v_x(x,y) = \frac{1}{2\pi}\int_0^1 \left\{T'(\xi)\frac{x-\xi}{d^2} \pm \alpha^2\left(\frac{M(\xi)-T'(\xi)}{1-\alpha^2}\right)\right.$$
$$\left.\times\left(k\,e^{k(x-\xi)}\left[\frac{x-\xi}{d}K_1(\pm kd) \pm K_0(\pm kd)\right] \mp \frac{x-\xi}{d^2}\right)\right\}d\xi. \tag{29}$$

Similar expressions may be written for all perturbation quantities.

Several properties of the solution are evident at this point. In the limit $R_m \to 0$, the solutions for $\mathbf{v}$ and $\mathbf{h}$ decouple to yield the conventional irrotational hydrodynamic and magnetostatic solutions, respectively, and, for $\alpha^2 = 0$, the solution for $\mathbf{v}$ reduces to the conventional solution, while the solution for $\mathbf{h}$ is irrelevant. Also, if $M = T'$, then $\mathbf{v}$ and $\mathbf{h}$ are again the conventional irrotational solutions, and $\mathbf{v} = \mathbf{h}$.

The situation for α^2 near unity requires some clarification. If α^2 approaches unity for any fixed values of R_m and d, then the product kd will approach zero. The small-argument expansions of K_0 and K_1 may then be employed in the rotational kernel together with an expansion of the exponentials, yielding the result

$$v_x(x,y) = \frac{1}{2\pi}\int_0^1 \left[T'(\xi)\left(\frac{x-\xi}{d^2}\right) - \tfrac{1}{2}R_m\alpha^2\{M(\xi)-T'(\xi)\}\left(\frac{y^2}{d^2}+\ln d\right)\right]d\xi\{1+O(k)\}. \tag{30}$$

Constant terms in the kernel do not contribute, since

$$\int_0^1 T'(x)\,dx = 0 = \int_0^1 M(x)\,dx.$$

In the same way, the vorticity is given for α^2 near 1 by

$$\omega_z(x,y) = \frac{R_m\alpha^2}{2\pi}\int_0^1 \{M(\xi)-T'(\xi)\}(yd^{-2})\,d\xi\{1+O(k)\}. \tag{31}$$

Consequently, small-perturbation solutions exist in the limit $\alpha^2 = 1$. These have finite current and vorticity proportional to R_m, in spite of the fact that f_1 and f_2 become large like $(1-\alpha^2)^{-1}$. This behaviour is found to be typical of the same limit in the lifting and slender-body problems as well, and indicates the possibility of rotational solutions with flow at the Alfvén speed. All of the above observations should be regarded as general properties of the small-perturbation flow for aligned fields.

It is interesting that the theory predicts well behaved solutions with flow at the Alfvén speed, in contrast to the results of Greenspan & Carrier (1959) for infinite conductivity and small viscosity, in which choking of the entire flow occurs for a finite flat plate at $\alpha^2 = 1$. As shown by Hasimoto (1959), the (apparent) convection term vanishes for $\alpha^2 = 1$ in the infinite-conductivity approximation, so that no non-trivial solution satisfying the no-slip condition on $\mathbf{V}$ can exist. On the other hand, for zero viscosity and finite conductivity, a proper solution for the flow appears to exist because a shear in $\mathbf{V}$ is permitted at the boundary, and $\mathbf{H}$ and $\mathbf{V}$ are not 'locked together' as with infinite conductivity.

A quantity of primary interest is the pressure at the airfoil surface. It is a familiar result from conventional thin-airfoil theory that d may be replaced in the kernel by $|x-\xi|$ in the consistent formula for flow quantities at the airfoil surface. It may be shown (Lary 1959) that, subject only to the limitations on ϵ and k derived above, the same is true of the rotational kernel.

Drag of thickness airfoils. It is possible at this point to calculate the forces acting on the thickness airfoil. Forces appear, due to the pressure acting on the airfoil surface and to the magnetic field acting on the system of magnets inside the airfoil, and lead to a drag D. The perturbation h_x is the same to first order in ϵ at $y = 0$ and $y = \pm\frac{1}{2}T(x)$ for $I = 0$. (This step is necessary, since perturbation quantities calculated from the singularity model are only appropriate as evaluated in the flow field, and not inside the body.) The consistent formulae for v_x and h_x at the airfoil surface are introduced, yielding

$$C_D = \frac{D}{\rho V_0^2 L} = \frac{R_m \alpha^2}{4\pi} \int_0^1 \int_0^1 F'(\xi)\, F'(x)\, e^{k(x-\xi)} \times \left(\frac{\pm(x-\xi)}{|x-\xi|} K_1(\pm k\,|x-\xi|) + K_0(\pm k\,|x-\xi|) \right) d\xi\, dx. \tag{32}$$

The 'effective thickness distribution' (in producing drag) has been denoted by F, where

$$F(x) = T(x) - \int_0^x M(\xi)\, d\xi. \tag{33}$$

The drag integral may be approximated for two cases: k small compared to unity, and k large compared to ϵ^{-1} (but still small compared to ϵ^{-2}). In the first case, the small argument expansions for K_0 and K_1 are used, leading to the result

$$C_D = -\tfrac{1}{4} R_m \alpha^2 \pi^{-1} \int_0^1 \int_0^1 F'(\xi)\, F'(x) \ln|x-\xi|\, d\xi\, dx\, \{1 + O(k)\}. \tag{34}$$

In the second case, the asymptotic forms for K_0 and K_1 may be employed, since the quantity $k\,|x-\xi|$ in their arguments represents kd, and d is of order ϵ or larger. Here, C_D has the form,

$$C_D = \pm \frac{\frac{1}{2}\alpha^2}{\pi} \frac{(\frac{1}{2}\pi\,|k|)^{\frac{1}{2}}}{1-\alpha^2} \int_0^1 \int_0^1 F'(\xi)\, F'(x) \frac{e^{k(x-\xi \mp |x-\xi|)}}{|x-\xi|^{\frac{1}{2}}} \left(\frac{\pm(x-\xi)}{|x-\xi|} + 1 \right) d\xi\, dx\, \{1 + O(k^{-1})\}. \tag{35}$$

The integrand vanishes for $\xi > x$ if $k > 0$ and for $\xi < x$ if $k < 0$, so that the kernel is simply $2\,|x-\xi|^{-\frac{1}{2}}$. In addition, the subcases $k > 0$ and $k < 0$ may be combined by noting that the simplified integrand is a symmetric function of x and ξ, or

$$C_D = \tfrac{1}{4}\alpha^2 \left(\frac{R_m}{\pi\,|1-\alpha^2|}\right)^{\frac{1}{2}} \int_0^1 \int_0^1 \frac{F'(\xi)\,F'(x)}{|x-\xi|^{\frac{1}{2}}} d\xi dx. \tag{36}$$

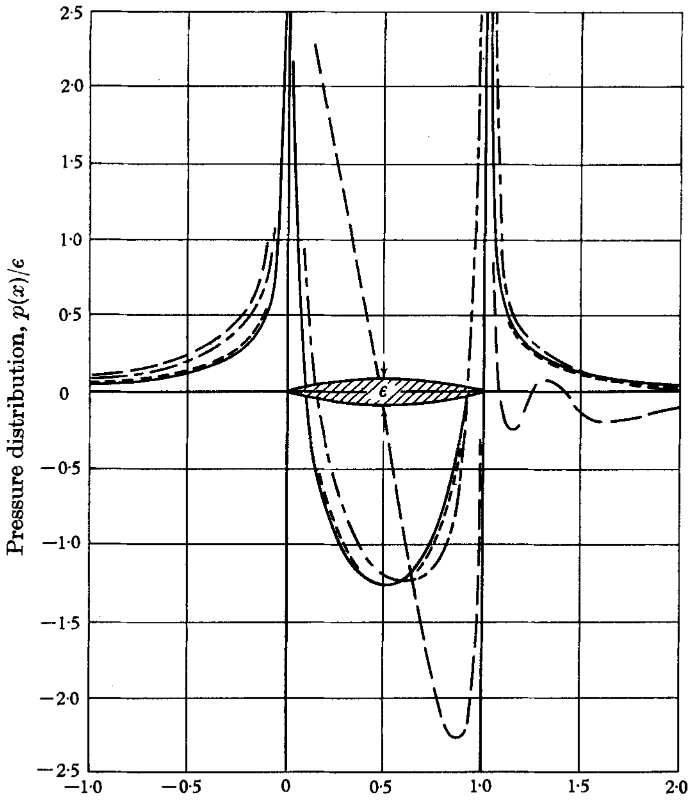

FIGURE 1. Pressure distribution on a parabolic-thickness airfoil and extended body axis for various values of R_m and for $\alpha^2 = \frac{1}{2}$. $R_m = 0$, ——; $R_m = 0{\cdot}4$, ----; $R_m = 4{\cdot}0$, — — —; $R_m = 40$, — —.

The drag of a thin airfoil is therefore proportional to ϵ^2 in either case, and to $R_m\alpha^2$ if k is small compared to unity, and $R_m^{\frac{1}{2}}\,|1-\alpha^2|^{-\frac{1}{2}}\,\alpha^2$ if k is between ϵ^{-1} and ϵ^{-2}. The pressure distribution and drag for a symmetrical airfoil of parabolic thickness distribution and thickness ratio ϵ are presented in figures 1 to 3. It is interesting that drag increases monotonically with α^2 for values of R_m less than about 10, but has a maximum near $\alpha^2 = 1$ for larger values of R_m. Dashed lines of constant k delineate régimes in which the various approximations to C_D are appropriate. The relative maximum of C_D for large values of R_m near $\alpha^2 = 1$ results from the increased penetration of the current layer into the fluid due to the near-zero effective convection speed of current and vorticity.

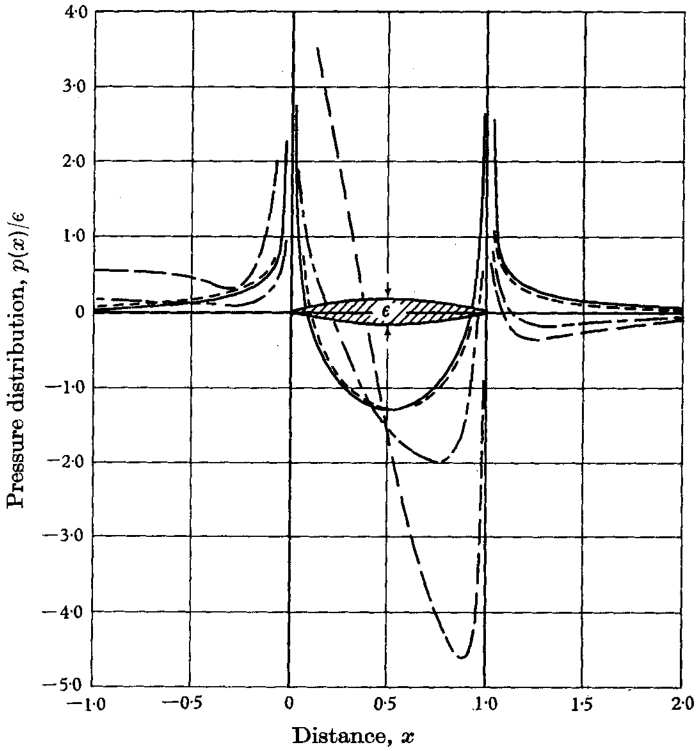

FIGURE 2. Pressure distribution on a parabolic-thickness airfoil and extended body axis for various values of R_m and for $\alpha^2 = \frac{3}{2}$. $R_m = 0$, ——; $R_m = 0·4$, ----; $R_m = 4·0$, — — —; $R_m = 40$, — —.

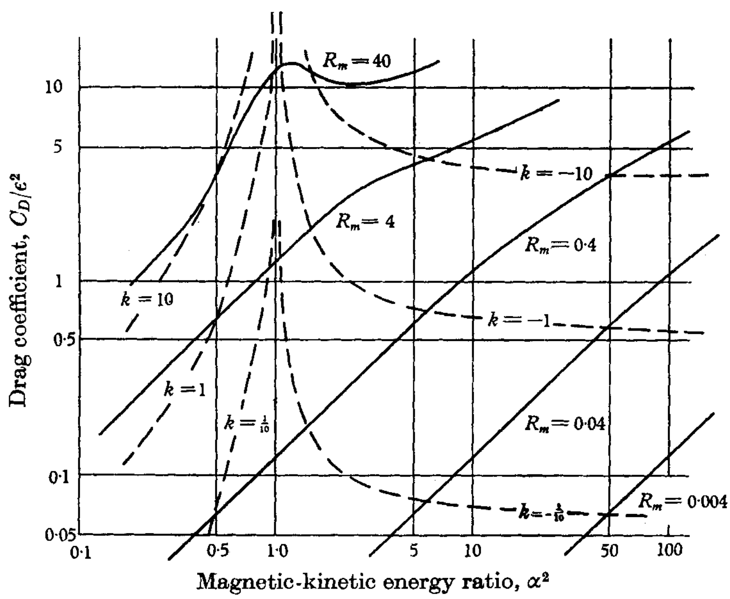

FIGURE 3. Drag coefficient of parabolic-thickness airfoil in a conducting fluid for several values of R_m. Lines of constant $k = \frac{1}{2}R_m(1-\alpha^2)$ delineate various régimes of flow.

The dependence of drag upon R_m and α^2 may also be deduced for the magnetic-boundary-layer limit, $|k| \gg \epsilon^{-2}$. The power DV_0 required to move the body through the fluid must be equal, from energy considerations, to the rate of dissipation of energy by ohmic heating. If one employs the results of Sears & Resler (1959), in which the magnetic-boundary-layer limit is characterized by a no-slip condition on **H**, then the current density is of order $H_0\sqrt{|k|/L}$ in the upper and lower layers. The result of the calculation of C_D is

$$C_D = O(|1-\alpha^2|^{\frac{1}{2}} R_m^{-\frac{1}{2}}\alpha^2). \tag{37}$$

The drag in this limit is analogous to the drag of viscous friction, in that both vary as the inverse square root of the appropriate Reynolds number and are essentially independent of ϵ for thin airfoils. In the case of the liquids mentioned earlier, for which the quantity R_m/R_ϵ is very small compared to unity, 'magnetic friction' plays a dominant role over viscous friction in determining the drag of the body, provided that R_m is larger than ϵ^{-2} and α^2 is not near zero.

The appearance of drag in the boundary-layer limit is in conflict with the prediction of McCune (1960) for the aligned-fields case. The above result indicates, moreover, that the departure of drag, for finite R_m, from the infinite conductivity value (of zero) is stronger in the aligned-fields case than in the crossed-fields case, in which the departure is proportional to R_m^{-1}.

Optimum drag bodies. The drag integral for $|k| \ll 1$ is the integral involved in the induced drag of finite wings in the Prandtl lifting-line theory and also in the Karman wave-resistance formula for supersonic bodies of revolution. Since the properties of this integral are well known, optimum drag bodies are easily calculated for this régime (Sears 1947). An example of the results is that the airfoil of minimum drag subject to a fixed cross-sectional area is an elliptical cylinder, provided that no magnetization is present. Similar techniques apply to the régime $1 \ll |k| \ll \epsilon^{-2}$, where the inner integral in equation (35) is the Abel integral, for which the general inversion is also well known (e.g. Whittaker & Watson 1952).

The lifting problem

A zero-thickness airfoil with ordinate $y = C(x)$ is located near the x-axis on the range $[0, 1]$ and contains a current distribution of strength $I(x)$, made non-dimensional by H_0. The boundary condition on **V** is satisfied by distributions g_1 and g_2 of the irrotational and rotational vortex singularities, respectively, or

$$v_y(x,y) = C'(x) = \frac{1}{2\pi}\int_0^1 \{g_1(\xi)\,(x-\xi)\,d^{-2} \pm \alpha^2 g_2(\xi)\,k\,e^{k(x-\xi)} \times [(x-\xi)\,d^{-1}K_1(\pm kd) \mp K_0(\pm kd)]\}\,d\xi. \tag{38}$$

The solution of the direct problem in conventional flow is possible only after some analysis (e.g. Glauert 1947) involving the particular nature of the kernel $(x-\xi)/d^2$ and the specification of a cyclic constant (circulation about the airfoil) by the Kutta condition. Presumably this analysis, which essentially seeks the 'resolvent kernel' of the integral equation involving the simplified kernel $(x-\xi)^{-1}$, could be duplicated for the present, more complicated kernel involving the modified Bessel functions. In view of the difficulties involved in this procedure, it would seem preferable, however, to resort to the device of studying first the mathematically direct problem of calculating the camber shape for a specified pressure loading G and current distribution I. Wherever possible, the problem of calculating G for a given airfoil shape will be solved.

Proceeding in this spirit, the discontinuity conditions on **H** and **V** yield equations in g_1 and g_2 similar to equations (22) and (24), except that the solutions

$$g_1(x) = \frac{G(x)-\alpha^2 I(x)}{1-\alpha^2}, \quad g_2(x) = \frac{I(x)-G(x)}{1-\alpha^2} \tag{39}$$

may be written directly with no limitation on ϵ or k. It must be recalled, however, that an approximation has been introduced in satisfying the boundary conditions at the axis rather than on the airfoil. This consideration requires that y again be regarded as a first-order quantity in ϵ, rather than zero, and leads to the same limitations on ϵ and k obtained in the analysis of the thickness problem.

Forces on the lifting airfoil. The lift $\mathscr{L}$ on the airfoil has contributions from the pressure difference across the airfoil and from the Lorentz force acting on the current inside the airfoil. The latter has the general form $\mathbf{J}\times\mathbf{B}$, so that the lift coefficient is simply

$$C_L = \frac{\mathscr{L}}{\rho V_0^2 L} = -\int_0^1 \{G(x)-\alpha^2 I(x)\}\,dx. \tag{40}$$

The drag coefficient,

$$C_D = \frac{R_m\alpha^2}{4\pi}\int_0^1\int_0^1 \tilde{F}(x)\,\tilde{F}(\xi)\,e^{k(x-\xi)}\left\{K_0(\pm k\,|x-\xi|) \mp \frac{(x-\xi)}{|x-\xi|}K_1(\pm k\,|x-\xi|)\right\}d\xi\,dx, \tag{41}$$

where $\tilde{F}(x) = G(x)-I(x)$, involves a slightly different kernel from its counterpart in the thickness problem.

Lifting airfoil of zero drag. A necessary and sufficient condition for zero-drag is $\tilde{F}=0$, the case of irrotational flow. The lift coefficient for the zero-drag case is denoted by $\bar{C}_L$ and is related to the conventional value $(C_L)_{\alpha=0}$ in the following way:

$$\bar{C}_L = -(1-\alpha^2)\int_0^1 G(x)\,dx = (1-\alpha^2)\,(C_L)_{\alpha=0}. \tag{42}$$

Equation (38) becomes
$$C'(x) = \frac{1}{2\pi}\int_0^1 \frac{G(\xi)}{x-\xi}\,d\xi, \tag{43}$$

so that the airfoil shape for a given G is the same as for the conventional case. The lift for a given airfoil shape is proportional to $(1-\alpha^2)$, provided that current is driven in the airfoil such as to maintain the equality of I and G. The results for this special case correspond quite closely to the infinite-conductivity results of Sears & Resler (1959). This includes the possibility of solving the mathematically indirect problem by inverting the integral equation in G (Glauert 1947).

It is possible to study some features of the general solution for a given airfoil shape by examining the limiting cases $|k| \ll 1$ and $|k| \gg 1$ for $I=0$. The inclusion of a known current distribution involves only a formal modification of the results.

The 'small k' approximation. In the first case, the small-argument expansions of K_0 and K_1 are used in equation (38) with the following result:

$$C'(x) = \frac{1}{2\pi}\int_0^1 \frac{G(\xi)}{x-\xi}d\xi - \frac{R_m\alpha^2}{4\pi}\int_0^1 (1+\gamma+\ln|\tfrac{1}{2}k|+\ln|x-\xi|)\,G(\xi)\,d\xi\,\{1+O(k)\}, \tag{44}$$

where γ is Euler's constant. The first term on the right-hand side is recognizable as the camber shape for irrotational flow. For a given airfoil shape, G may be approximated for small values of $R_m\alpha^2$ by employing the conventional ($R_m = 0$) solution G_0, say, then solving the following iterated equation for G:

$$\frac{1}{2\pi}\int_0^1 \frac{G(\xi)}{x-\xi}d\xi = C'(x) + \frac{R_m\alpha^2}{4\pi}\int_0^1 (1+\gamma+\ln|\tfrac{1}{2}k|+\ln|x-\xi|)\,G_0(\xi)\,d\xi. \tag{45}$$

It may be shown (Lary 1959) in this way that the loading for a flat plate at incidence A_0 also contains a term associated with parabolic camber for irrotational flow. The lift for this case is

$$C_L = \pi A_0[1+\tfrac{1}{4}R_m\alpha^2(1+\gamma+\ln|\tfrac{1}{2}k|+\tfrac{1}{2})]. \tag{46}$$

The above result is not valid if k is very near zero, since the term involving $\ln|\tfrac{1}{2}k|$ is no longer small. However, for fixed R_m and loading G, the limit $\alpha^2 \to 1$ (i.e. $k \to 0$) may be studied by retaining on the right in equation (44) only the leading term in $\ln|k|$, or

$$C'(x) = -\tfrac{1}{4}R_m\alpha^2\pi^{-1}\ln|k|\int_0^1 G(x)\,dx + O(G). \tag{47}$$

This equation may be written directly as the lift formula for a flat plate, i.e. as

$$C_L = -\int_0^1 G(\xi)\,d\xi = \frac{A_0}{\tfrac{1}{4}R_m\alpha^2\pi^{-1}\ln|k^{-1}|+O(1)}. \tag{48}$$

A particularly interesting feature of the lift curve for the flat plate is that C_L has an extremum at $\alpha^2 = 1$, where $C_L = 0$, and is positive on either side of $\alpha^2 = 1$ for positive incidence. This behaviour is shown in figure 4.

The 'large k' approximation. In the second case, $|k| \gg 1$, the asymptotic form of K_0 reveals that, for $k > 0$, the contribution of the rotational term for $\xi > x$ in equation (38) is dominated by the exponential $e^{-2k(\xi-x)}$, and may be neglected in the limit of large k. (This contribution is of order $|k|^{-1}$, whereas the range from 0 to x contributes a term of order $|k|^{-\frac{1}{2}}$.) An appropriate formula, valid to within terms of order $1/k$, is given by integrating equation (38) by parts, then differentiating with respect to x. For $k > 0$, this has the form

$$2\pi(1-\alpha^2)\,C'(x) = \int_0^1 \frac{G(\xi)}{x-\xi}d\xi + \alpha^2\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^x \frac{G(\xi)}{(x-\xi)^{\frac{1}{2}}}d\xi. \tag{49}$$

A similar expression may be written for $k < 0$, where the integration limits on the last term are x and 1.

Equation (49) suggests that, for $|k| \gg 1$, the terms of order $|k|^{-\frac{1}{2}}$ may be excluded in a first approximation to the solutions. (It should be recalled in this connexion that the bound on k permits passage to arbitrarily large values provided that ϵ is made sufficiently small. The term 'magnetic boundary layer' has, however, been reserved herein for the case $|k| \gg \epsilon^{-2}$, in which the layer is thinner than the body and a no-slip condition on $\mathbf{H}$ results.) In this approximation, the solution is again obtainable from the results of the conventional theory and is identical to the predictions of the magnetic-boundary-layer theory of Sears & Resler (except for the specification of the cyclic constant, which is discussed in the subsequent section). The results are obtained through radically different approximations to the boundary condition on $\mathbf{H}$ (appropriate to the respective conductivity range), and the agreement indicates the consistency of the approximations made in either case.

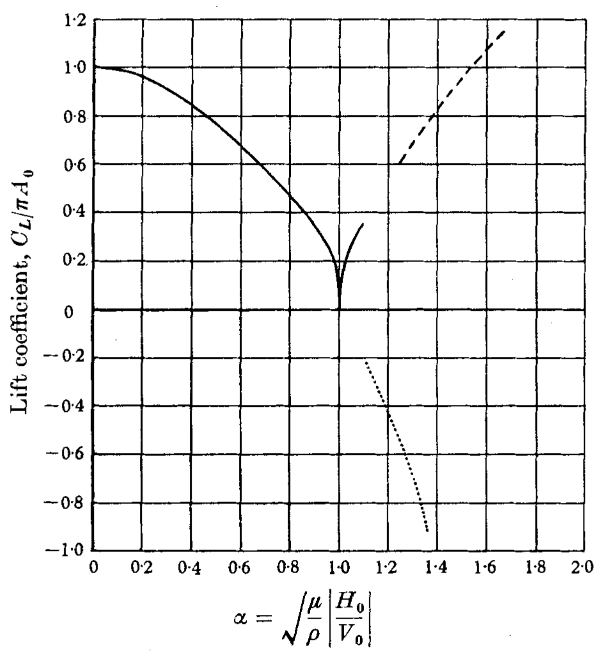

FIGURE 4. Lift coefficient C_L of a flat plate at incidence A_0 for $R_m = 10$. Calculated using Kutta condition, ...; calculated using magnetohydrodynamic analogue to Kutta condition, -------.

The Kutta condition and a magnetohydrodynamic analogue. The analogy of the effects of finite conductivity in magnetohydrodynamic flow with aligned fields to the effects of non-zero viscosity in the conventional flow is apparent, since both produce wake regions of rotational flow which degenerate to boundary layers in the limit of large values of the appropriate (viscous or magnetic) Reynolds number. (The similarity even extends in the Sears–Resler theory to the no-slip condition on $\mathbf{H}$ in the magnetic-boundary-layer limit.) The question therefore arises as to what, if any, analogue to the Kutta condition arises from using the approximation of large R_m in solving the magnetohydrodynamic flow about thin airfoils.

Howarth (1935) demonstrated that the Kutta condition evolves from the fact that, for steady flow, no net vorticity is shed into the wake. It is, therefore, the fact that vorticity is convected into the wake from the trailing edge that selects this edge as the one from which the velocity singularity must be removed in the boundary-layer limit. From this consideration, one might conjecture that the Kutta condition should be replaced by a condition which removes any velocity or magnetic field singularity which occurs at the edge from which the vorticity and current flow into the wake. (It has been shown above that the magneto-hydrodynamic wake lies downstream of the body for $\alpha^2 < 1$ and upstream for $\alpha^2 > 1$.) Some evidence in favour of this conjecture is obtained within the framework of the small perturbation theory for the 'large k' range of conductivity. In particular, it may be shown (Lary 1959) that the condition predicted by the above reasoning is necessary for the existence of a solution to the approximate integral equation (i.e. equation (49) or its analogue for $k < 0$).

A consequence of the change of end condition on G at $\alpha^2 = 1$ is that the lift of a flat plate at incidence in the infinite-conductivity approximation should not change sign (as originally predicted in the Sears–Resler theory) at $\alpha^2 = 1$, but should always be positive (or zero) for positive incidence. In this connexion, it is shown directly in equation (48) that the lift of a flat plate vanishes at $\alpha^2 = 1$, but is positive for positive incidence on either side of $\alpha^2 = 1$ for arbitrary, but finite, conductivity. Since lift is proportional to the circulation, this provides further evidence that magnetic effects influence the determination of the cyclic constant. ('Cyclic constant' here implies the circulation at infinity, since the flow may be rotational in a large region near the body.)

An exception to the altered Kutta condition is the case of irrotational flow, $I = G$, in which no magnetohydrodynamic interaction occurs. On the other hand, for the infinite-conductivity limit, the Sears–Resler theory indicates the equality of $\mathbf{h}$ and $\mathbf{v}$ in the flow, and thus of the irrotational source and/or vortex distributions. G is determined uniquely by C' (and the appropriate leading- or trailing-edge condition) so that I should not be interpreted as the current being driven in the airfoil, which may be an insulator, but rather as the net current flowing in the airfoil and in the upper and lower boundary layers. In this latter case the altered condition is again necessary.

The effect of viscosity. With regard to the magnetic-boundary-layer limit, it must be recalled that viscous effects are present in this case as well. This involves no essential complication for $\alpha^2 < 1$, since the Kutta condition coincides with the magnetohydrodynamic condition. However, for $\alpha^2 > 1$, the two conditions are not generally compatible. A choice must be made, then, between viscous and magnetohydrodynamic considerations as a basis for determining the cyclic constant. In the case where the viscous boundary layer lies inside the magnetic boundary layer, the Howarth argument regarding vorticity flux into the viscous wake is not valid, since vorticity is convected into the magnetic wake as well. (The supposition that the boundary layers decouple is based on the fact that $|k|/R_e$ is greatly different from unity, so that viscous and magnetic effects then occur with considerably different length scales.) This admits the possibility of an altered condition on the velocity discontinuity, such as that conjectured above.

For the case where the viscous boundary layer is the outer layer, Hasimoto (1959) has shown that the reversed wake again appears. Hasimoto employs the concept of an effective reversal of stream direction for $\alpha^2 > 1$, and thus implicitly replaces the usual Kutta condition for the limit of large R_e. Thus, with the inclusion of the Hasimoto observation, it might further be conjectured that the altered Kutta condition applies to large, as well as small, values of the ratio $|k|/R_e$.

Slender body theory

A slender body of revolution with cross-sectional area $S(x)$ is located on the x-axis and contains magnetization $M(x)$, made non-dimensional by H_0L. As in the airfoil theory, the approximate solutions for the source distributions,

$$\hat{f}_1(x) = \frac{S'(x)-\alpha^2 M(x)}{1-\alpha^2}, \quad \hat{f}_2(x) = \frac{M(x)-S'(x)}{1-\alpha^2}, \tag{50}$$

may be obtained subject to $\qquad |k| \ll 1/\epsilon^2 \ln \epsilon. \qquad$ (51)

The restriction on k arises from the fact that the rotational singularities are distributed on the x-axis in order to satisfy boundary conditions on the body surface, a distance ϵ away. For this reason it might be expected that the range of k could be extended by distributing singularities very near the body surface, on rings concentric with the axis, for example. Applied to the irrotational source, this technique leads to functions which involve complete elliptic integrals, whereas more complicated functions are obtained in the case of the rotational source. Near the body these functions are asymptotic for large $|k|$ to the solutions for the plane-flow rotational source derived previously. It is therefore likely that this technique would in fact extend the range of k in which a solution could be obtained to that corresponding to the airfoil flows, but the mathematical complications are severe.

All perturbation quantities may be evaluated directly by integration over the known singularity distributions, using the appropriate kernels. As an example of the form of the integrals, v_x may be written in the usual way as

$$v_x(x,r) = \frac{1}{4\pi}\int_0^1 \left\{S'(\xi)\frac{(x-\xi)}{s^3} + \alpha^2\left(\frac{M(\xi)-S'(\xi)}{1-\alpha^2}\right)\right.$$
$$\left.\times\left[\frac{(x-\xi)}{s^3}\left(e^{k(x-\xi\mp s)}-1\right)+\frac{k}{s}\left(1\pm\frac{(x-\xi)}{s}\right)e^{k(x-\xi\mp s)}\right]\right\}d\xi. \tag{52}$$

The drag is calculated in the same way as for the thickness-airfoil problem, *with the result that*

$$C_D = \frac{1}{4\pi}\frac{\alpha^2}{(1-\alpha^2)}\int_0^1\int_0^1 \left\{[S'(\xi)-M(\xi)][S'(x)-M(x)]\right.$$
$$\left.\times\left[\frac{(x-\xi)}{s^3}+\frac{k}{s}\left(1\pm\frac{(x-\xi)}{s}\right)\right]e^{k(x-\xi\mp s)}\right\}d\xi\,dx. \tag{53}$$

Because of the nature of the axisymmetric solution, it is not possible to approximate quantities on the body surface by the corresponding values on the body axis. For this reason s may not be replaced by $|x-\xi|$ in equation (53), for example, since r must be regarded as $R(x)$.

The results for $|k| \ll 1$ may, however, be expressed by expanding the exponential involved in the respective kernels. Equation (52) may then be written as a formula for pressure at the body surface.

$$p\{x, R(x)\} = -\frac{1}{4\pi}\int_0^1 \left[S'(\xi)\frac{(x-\xi)}{s^3} + \tfrac{1}{2}R_m\alpha^2\{M(\xi)-S'(\xi)\} \times \left(\frac{2(x-\xi)^2+R^2(x)}{s^3}\right)\right] d\xi\,\{1+O(k)\}. \quad (54)$$

The same approximation, applied to equation (53), leads to the following drag formula for $|k| \ll 1$:

$$C_D = \frac{R_m\alpha^2}{8\pi}\int_0^1\int_0^1 \{S'(\xi)-M(\xi)\}\{S'(x)-M(x)\}\left(\frac{2(x-\xi)^2+R^2(x)}{s^3}\right) d\xi\, dx. \quad (55)$$

It is evident from equation (52) that the task of approximating results for $|k| \gg 1$ is not as straightforward as in the airfoil theory.

Conclusion

The finite-conductivity solutions obtained above for small perturbations to uniform aligned magnetic and velocity fields satisfy the Oseen equation of viscous-flow theory, but do not suffer with the viscous case the defect of having boundary conditions which are inconsistent with the linearization of the equations.

The flow is found to be essentially irrotational at large distances, except in a parabolic wake which extends either downstream or upstream from the body, depending on whether the stream speed is greater or less than the Alfvén speed. The direction of the wake appears, for large values of the parameter $|k|$, to influence the choice of the cyclic constant (circulation at infinity) for the lifting-airfoil flows. In any case, the current and vorticity generated by the fluid-field interaction lead to a drag which is proportional to the magnetic energy density and increases like R_m for $|k| \ll 1$ and like $R_m^{\frac{1}{2}}\,|1-\alpha^2|^{-\frac{1}{2}}$ for $\epsilon^{-1} \ll |k| \ll \epsilon^{-2}$.

The question of stability of the flow solutions, especially as regards the unfamiliar upstream-wake behaviour, has not been considered. Such considerations may play an important role in resolving the question of the proper (Kutta-type) condition for lifting airfoils, as well as the significance of solutions which involve large perturbations at infinity.

The author is greatly indebted to Professor W. R. Sears and to Professor E. L. Resler, who suggested the present research, for many helpful discussions and valuable suggestions.

REFERENCES

Glauert, H. 1947 *The Elements of Aerofoil Theory*. Cambridge University Press.

Greenspan, H. P. & Carrier, G. F. 1959 The magnetohydrodynamic flow past a flat plate. *J. Fluid Mech.* **6**, 77.

Hasimoto, H. 1959 Viscous flow of a perfectly conducting fluid with a frozen magnetic field. *Phys. Fluids*, **2**, 337.

HOWARTH, L. 1935 The theoretical determination of the lift coefficient for a thin elliptic cylinder. *Proc. Roy. Soc.* A, **149**, 558.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LARY, E. C. 1959 A theory of thin airfoils and slender bodies in fluids of arbitrary electrical conductivity. Ph.D. Thesis, Cornell University.

MCCUNE, J. E. 1960 On the motion of thin airfoils in fluids of finite electrical conductivity. *J. Fluid Mech.* **7**, 449.

SEARS, W. R. 1947 On projectiles of minimum wave drag. *Quart. Appl. Math.* **4**, 361.

SEARS, W. R. 1959 Magnetohydrodynamic effects in aerodynamic flows. *ARS Journal*, **29**, 397.

SEARS, W. R. & RESLER, E. L. 1959 Theory of thin airfoils in fluids of high electrical conductivity. *J. Fluid Mech.* **5**, 257.

STEWARTSON, K. 1960 On the motion of a non-conducting body through a perfectly conducting fluid. *J. Fluid Mech.* **8**, 82.

WHITTAKER, E. T. & WATSON, G. N. 1952 *A Course of Modern Analysis*. Cambridge University Press.